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RECOGNITION OF MATRIX RINGS II

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The work in this paper was under active discussion with Shimshon Amitsur at the time of his death, and there was already agreement to prepare this joint paper. We are saddened by the loss of a colleague and wise friend, but are pleased to be able to remember him.

ABSTRACT

This paper provides several new criteria for a ring to be a complete matrix ring. Some applications demonstrate their efficacy; and their relative strengths are indicated by calculating the structures they impose on universal algebras.

Introduction

Stimulated by a question of Chatters [4], several recent papers, such as Chatters [5], Levy, Robson and Stafford [7] and Robson [8], have investigated techniques enabling identification of complete $n \times n$ matrix rings. This paper is a continuation and improvement of [8], in which it was shown, for a ring R, that the following conditions are equivalent:

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(i) R is an $n \times n$ matrix ring;

(ii) R contains elements f, a such that $f^n = 0$ and

$$af^{n-1} + faf^{n-2} + \cdots + f^{n-1}a = 1;$$

(iii) R contains elements $f, a_1, a_2, \ldots, a_n$ such that $f^n = 0$ and

$$a_1 f^{n-1} + f a_2 f^{n-2} + \cdots + f^{n-1} a_n = 1.$$

The first section of this paper starts by providing two further criteria each involving three elements and each a substantial simplification of (ii) and (iii); namely

(iv) R contains elements f, a, b such that $f^n = 0$ and $af^{n-1} + fb = 1$;

(v) R contains elements f, a, b such that $f^n = 0$ and $af^M + f^N b = 1$, for some M, N with $M + N = n$.

The section ends with a description of an algebra universal with respect to criterion (iv). Section 2 then uses these results to consider criteria similar to (iv) and (v) but, like (ii), involving only two elements. Here the situation is more complex and the theory is less complete. The final section gives further applications, this time to rings of differential operators.

Throughout, all rings are associative and, except when stated otherwise, have a 1. The standard matrix units of an $n \times n$ matrix ring $M_n(S)$ over some ring S will be denoted by $\{e_{ij}\}$. On the other hand, when displaying a family of elements satisfying the relations characterizing matrix units, we will write these as $\{E_{ij}\}$.

1. Three element relations

This section concerns relations involving three elements of a ring R. The first result establishes one of the new criteria mentioned in the introduction.

THEOREM 1.1: *The following conditions on a ring R are equivalent:*

(i) *R is a complete $n \times n$ matrix ring;*

(ii) *R contains elements a, b and f such that $f^n = 0$ and $1 = af^{n-1} + fb$.* ∎

Proof: (i) $\Rightarrow$ (ii). Let $\{e_{ij}\}$ be a complete set of $n \times n$ matrix units for R. We let $a = e_{1n}$, $b = e_{12} + e_{23} + \cdots + e_{n-1,n}$ and $f = e_{21} + e_{32} + \cdots + e_{n,n-1}$. One can then verify that $f^{n-1} = e_{n1}$, that $f^n = 0$ and that $af^{n-1} + fb = 1$, as required.

(ii) $\Rightarrow$ (i). The argument here concentrates on the n right ideals $f^r af^{n-1}R$ for $r \in \{0, 1, \ldots, n-1\}$. It will be shown that these right ideals are mutually isomorphic, and that their sum is direct and equals R. Granted this, the regular representation then demonstrates that $R \simeq M_n\ (\mathrm{End}(af^{n-1}R))$.

We start, then, by noting that

$$af^{n-1} = af^{n-1}(af^{n-1} + fb) = (af^{n-1})^2$$

since $f^n = 0$. It follows that, for each $r \in \{0, 1, \ldots, n-1\}$, the two maps $af^{n-1}R \to f^r af^{n-1}R$ and $f^r af^{n-1}R \to af^{n-1}R$ given by left multiplication, respectively, by f^r and af^{n-1-r} are mutually inverse. This shows that the right ideals are isomorphic.

Next note that

$$\begin{aligned} 1 &= af^{n-1} + fb \\ &= af^{n-1} + f(af^{n-1} + fb)b \\ &= af^{n-1} + faf^{n-1}b + f^2(af^{n-1} + fb)b^2 \\ &= \ldots \\ &= af^{n-1} + faf^{n-1}b + \cdots + f^{n-1}af^{n-1}b^{n-1} \end{aligned} \tag{1}$$

since $f^n = 0$. Therefore $\sum_{r=0}^{n-1} f^r af^{n-1}R = R$.

Finally, to see that this sum is direct, suppose that

$$0 = af^{n-1}x_0 + faf^{n-1}x_1 + \cdots + f^{n-1}af^{n-1}x_{n-1}$$

for some $x_i \in R$. Left multiplication, in turn, by $af^{n-1}, faf^{n-2}, \ldots, f^{n-1}a$ shows that, for each r, $f^r af^{n-1}x_r = 0$. This is the final ingredient required. ∎

Note 1.2: For what follows, it is useful to observe, from the above proof, that left multiplication by f gives the isomorphism $f^{r-1}af^{n-1}R \to f^r af^{n-1}R$ for each $r \in \{1, 2, \ldots, n-1\}$ and it maps $f^{n-1}af^{n-1}R$ to zero. Thus, under the regular representation, $f = e_{21} + e_{32} + \cdots + e_{n,n-1}$. Likewise $af^{n-1} = e_{11}$ and so $f^r af^{n-1} = e_{r1}$ and $fb = 1 - e_{11}$. In fact, as the next result shows, one can describe, directly, the complete set of matrix units in terms of a, b and f, thereby also providing an alternative proof of Theorem 1.1.

THEOREM 1.3: *Let R be a ring containing elements f, a, b such that $f^n = 0$ and $af^{n-1} + fb = 1$. Then the set $\{E_{ij}\}$, given by $E_{ij} = f^{i-1}af^{n-1}b^{j-1}$, is a complete set of $n \times n$ matrix units for R.*

Proof: Note first that

$$\begin{aligned}&E_{11} + E_{22} + \cdots + E_{nn}\\ &= af^{n-1} + faf^{n-1}b + \cdots + f^{n-1}af^{n-1}b^{n-1}\\ &= 1\end{aligned}$$

as in (1).

So it remains only to show that $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$; i.e. that

$$(f^{i-1}af^{n-1}b^{j-1})(f^{k-1}af^{n-1}b^{\ell-1}) = \delta_{jk}(f^{i-1}af^{n-1}b^{\ell-1}).$$

Therefore it will be enough to prove that

$$af^{n-1}b^{j-1}f^{k-1}af^{n-1} = \delta_{jk}af^{n-1} \tag{2}$$

for all $j, k \in \{1, 2, \ldots, n\}$.

The proof of this starts by noting that

$$\begin{aligned}f^ib^i &= f^{i-1}(fb)b^{i-1} = f^{i-1}(1 - af^{n-1})b^{i-1}\\ &= (1 - f^{i-1}af^{n-i})f^{i-1}b^{i-1}\end{aligned}$$

for $1 \leq i \leq n$. Hence, using induction on i,

$$f^ib^i = (1 - f^{i-1}af^{n-i})(1 - f^{i-2}af^{n-i+1})\cdots(1 - af^{n-1}).$$

Using this, we see that

$$\begin{aligned}&af^{n-1}b^{j-1}f^{k-1}af^{n-1} = af^{n-j}(f^{j-1}b^{j-1})f^{k-1}af^{n-1}\\ &= af^{n-j}(1 - f^{j-2}af^{n-j+1})(1 - f^{j-3}af^{n-j+2})\cdots(1 - af^{n-1})f^{k-1}af^{n-1}.\end{aligned} \tag{3}$$

We will simplify this last expression, starting at its right-hand end, using the fact that

$$(1 - f^{n-\ell+1}af^{\ell})f^{k-1} = f^{k-1}$$

whenever $k-1+\ell \geq n$, since $f^n = 0$. First, suppose that $j \leq k$. Then (3) reduces to the equation

$$af^{n-1}b^{j-1}f^{k-1}af^{n-1} = af^{n-j}f^{k-1}af^{n-1}$$

which, since $f^n = 0$ and $(af^{n-1})^2 = af^{n-1}$, verifies (2) in this case. Next, suppose that $j > k$. Then the simplification of (3) produces the equation

$$af^{n-1}b^{j-1}f^{k-1}af^{n-1} = af^{n-j}(1-f^{j-2}af^{n-j+1})\cdots(1-f^{k-1}af^{n-k})f^{k-1}af^{n-1}.$$

However

$$(1-f^{k-1}af^{n-k})f^{k-1}af^{n-1} = f^{k-1}(af^{n-1}-(af^{n-1})^2) = 0.$$

Hence (2) holds for all j, k. ∎

Note 1.4: In fact, the set $\{E_{ij}\}$ obtained in 1.3 coincides with the set $\{e_{ij}\}$ arising from the regular representation in 1.1(ii). To see this, one notes, from 1.2 and 1.3, that $E_{i1} = e_{i1}$ for each i. However, each matrix unit is characterized precisely by its action, via left multiplication, on the set of right ideals $\{e_{i1}R : \ i = 1, \ldots, n\}$. Hence the two sets coincide.

Next we aim at the second of the new criteria. It is convenient first to prove two subsidiary results.

LEMMA 1.5: *Suppose that R is a ring, $f \in R$ and I, J are subrings of R, not necessarily with 1, such that J is closed under left and right multiplication by f. If $1 \in If + fJ$ then $1 \in If^2 + J$.*

Proof: Suppose that $1 = af + fb$ where $a \in I$, $b \in J$. Then

$$\begin{aligned} 1 &= af + fb \\ &= a(1-fb)f + fb + bf - (1-af)bf \\ &= a(af)f + (fb + bf - (fb)bf) \\ &\in If^2 + J. \end{aligned}$$ ∎

COROLLARY 1.6: *Suppose that R is a ring, $f \in R$ and N is a positive integer. Then*

$$\begin{aligned} & 1 \in Rf + f^{N-1}R \\ \Longleftrightarrow\quad & 1 \in Rf^2 + f^{N-2}R \\ & \dots \\ \Longleftrightarrow\quad & 1 \in Rf^i + f^{N-i}R \\ & \dots \\ \Longleftrightarrow\quad & 1 \in Rf^{N-1} + fR. \end{aligned}$$

Proof: For each $i \in \{1, 2, \dots, N-1\}$, both Rf^{i-1} and $f^{N-i-1}R$ are subrings closed under left and right multiplication by f. The result follows using Lemma 1.5, together with its left-right symmetric version. ∎

The next result establishes the second new criterion, which provides greater symmetry in the roles of a and b.

THEOREM 1.7: *For a ring R and positive integers m and n, the following conditions are equivalent:*

(i) *$R \simeq M_{m+n}(S)$ for some ring S;*
(ii) *R contains elements a, b and f such that $f^{m+n} = 0$ and $af^m + f^n b = 1$.*

Proof: To prove (i) $\Rightarrow$ (ii) note that the elements

$$\begin{aligned} a &= e_{1,m+1} + e_{2,m+2} + \cdots + e_{n,m+n} \\ b &= e_{1,n+1} + e_{2,n+2} + \cdots + e_{m,m+n} \\ f &= e_{21} + e_{32} + \cdots + e_{m+n,m+n-1} \end{aligned}$$

in $M_{m+n}(S)$ satisfy the equations in (ii).

Conversely, assume $a, b, f \in R$ are as described in (ii). Since $1 \in Rf^m + f^n R$ then Corollary 1.6 asserts that $1 \in Rf^{m+n-1} + fR$. Then Theorem 1.1 shows that $R \simeq M_{m+n}(S)$ for some ring S. ∎

We end this section by considering a universal example of R as in Theorem 1.1. Here and later, it will be useful to consider not only algebras over a commutative ring k but, more generally, over a noncommutative ring k, with the understanding that all generators are k-centralizing; i.e. commute with all elements of k.

THEOREM 1.8: *Let R be an algebra which is freely generated over some non-commutative ring k by three elements a, b and f subject only to the relations $f^n = 0$ and $1 = af^{n-1} + fb$. Then $R \simeq M_n(S)$ with S being a free k-algebra in n^2 indeterminates.*

Proof: One knows from Theorem 1.1 that R is a full ring of $n \times n$ matrices and that f, af^{n-1} and fb are as specified in Note 1.2. The conditions that these requirements lay upon the entries of a and b, when viewed as $n \times n$ matrices, specify that

$$a = \begin{pmatrix} * & \dots & * & 1 \\ * & \dots & * & 0 \\ & \dots & & \vdots \\ * & \dots & * & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ * & * & * & \dots & * \end{pmatrix}$$

where $*$ denotes arbitrary elements. To see that $R \simeq M_n(S)$ with S of the form claimed we argue as follows.

In $M_n(k\langle x_1, \dots, x_{n^2}\rangle)$ one can construct two matrices a' and b' of the same form as a and b above, with the $*$ entries being filled by the n^2 indeterminates x_i. There is thus a mapping

$$\beta\colon k\langle a', b', f\rangle \to R$$

given by $\beta(a') = a$, $\beta(b') = b$, $\beta(f) = f$. However, since a', b' and f satisfy the relations imposed on a, b and f, then β has an inverse. So $R \simeq k\langle a', b', f\rangle$. One can verify easily that the elements $a', b', f \in M_n(k\langle x_1, \dots, x_{n^2}\rangle)$ generate all the matrix units of $M_n(k\langle x_1, \dots, x_{n^2}\rangle)$. Hence $k\langle a', b', f\rangle$ contains the complete set of $n \times n$ matrix units. Since the indeterminates all appear as entries of the matrices a' and b', one sees that $k\langle a', b', f\rangle = M_n(k\langle x_1, \dots, x_{n^2}\rangle)$ as required. ∎

Comment: (i) The $n \times n$ matrix units which Theorem 1.3 provides in the ring $k\langle a', b', f\rangle$ coincide with the standard matrix units of the ring $M_n(k\langle x_1, \dots, x_{n^2}\rangle)$. This follows, as in Note 1.4, since one can readily verify that this is true for the first column of these sets.

(ii) We have not yet determined the corresponding result for the two relations $f^{m+n} = 0$ and $af^m + f^n b = 1$.

2. Two element relations

Comparison of the new criteria, in Theorem 1.1 and Theorem 1.7, with the criterion (ii) in the theorem of Robson [8] described in the introduction, leads to some natural conjectures about the existence of similar criteria involving only two elements. The next result disposes of the simplest guess. We thank the referee for this shorter proof.

THEOREM 2.1: *Let $n \geq 3$. Then there is no nontrivial ring R having elements a and f with $f^n = 0$ and $af^{n-1} + fa = 1$.*

Proof: Let R be such a ring. Multiply the equation $af^{n-1} + fa = 1$ on the left by f^{n-2} and on the right by f^{n-1}. This yields the equation

$$f^{n-1}af^{n-1} = f^{2n-3} = 0$$

which by Theorem 1.3 implies that $E_{n1} = 0$ and so that R is trivial. ∎

A similar argument shows the same for the relations $f^{m+n} = 0$ and $af^m + f^n a = 1$ whenever $m \neq n$. A more general result than this, replacing the relation $f^{m+n} = 0$ by $f^\nu = 0$ for some ν, can be found in Agnarsson [1].

However, there are some interesting two element criteria. If one considers the explicit choices of f, a and b in the proof that (i) $\Rightarrow$ (ii) in Theorem 1.1, one observes that $b^{n-1} = a$ and that $b^n = 0$. Therefore the next result follows directly from Theorem 1.1.

THEOREM 2.2: *The following conditions on a ring R are equivalent:*
- (i) *R is a complete $n \times n$ matrix ring;*
- (ii) *R contains elements b and f such that $f^n = 0$ and $1 = b^{n-1}f^{n-1} + fb$;*
- (iii) *R contains elements b and f such that $f^n = 0$, $b^n = 0$ and $1 = b^{n-1}f^{n-1} + fb$.* ∎

In the case of criterion (iii) above, it is particularly easy to describe the ring over which R is an $n \times n$ matrix ring.

COROLLARY 2.3: *Let R be a ring containing elements b, f with $b^n = f^n = 0$ and $b^{n-1}f^{n-1} + fb = 1$. Then $R \simeq M_n(b^{n-1}Rf^{n-1}) \simeq M_n(f^{n-1}Rb^{n-1})$.*

Proof: Note, from Theorem 1.3, that $E_{ij} = f^{i-1}b^{n-1}f^{n-1}b^{j-1}$ defines a set of matrix units. In particular

$$E_{1n} = b^{n-1}f^{n-1}b^{n-1} = (1 - fb)b^{n-1} = b^{n-1}$$

and similarly $E_{n1} = f^{n-1}$. The well known fact that for all i,j there is an isomorphism $R \simeq M_n(E_{ij}RE_{ji})$, where $E_{ij}RE_{ji}$ is a ring with unit E_{ii}, now gives the result. ∎

One can also obtain a result which, like Theorem 1.7, has more symmetry.

THEOREM 2.4: *Let i,j,n be integers with $i+j=n$. The following conditions on a ring R are equivalent:*

(i) *R is a complete $n \times n$ matrix ring;*
(ii) *R contains elements b and f such that $f^n = 0$ and $1 = b^i f^i + f^j b^j$;*
(iii) *R contains elements b and f such that $f^n = 0$, $b^n = 0$ and $1 = b^i f^i + f^j b^j$.*

Proof: To see that (i) implies the other two conditions, we simply use the same elements f and b as in Theorem 2.2 (i.e. those used in the proof that (i) $\Rightarrow$ (ii) in Theorem 1.1). The reverse implications follow from Theorem 1.7. ∎

Our next result, which concerns k-algebras of the type described before Theorem 1.8, shows that there are other relations on f and b which do not produce triviality. Hence, the relations described in Theorem 2.4 do not exhaust the possibilities.

THEOREM 2.5: *Let k be a noncommutative ring and i,j,m,n be positive integers. Let R be the k-algebra freely generated by elements b and f subject to the relations $f^{m+n} = 0$ and $b^i f^m + f^n b^j = 1$. Suppose that $i/m = j/n$. Then R is a non-trivial $(m+n) \times (m+n)$ matrix ring.*

Proof: That $R \simeq M_{m+n}(S)$ for some ring S is clear from Theorem 1.7. To establish non-triviality, first let d be any positive integer and define s and t in $M_d(k)$ by

$$s = e_{12} + e_{23} + \cdots + e_{d-1,d} \quad \text{and} \quad t = e_{21} + e_{32} + \cdots + e_{d,d-1} + e_{1d}.$$

Of course, these centralize k and one can readily check that $s^d = 0$, that $t^d = 1$, and that $t^j s^j + s^{d-j} t^{d-j} = 1$ for all $j \in \{1, 2, \ldots, d-1\}$.

Next, suppose that $i/m = j/n = p/q$ and choose $d = (m+n)p$. We set $b = t^q$ and $f = s^p$ in $M_d(k)$. Evidently $f^{m+n} = s^d = 0$ and

$$\begin{aligned} b^i f^m + f^n b^j &= t^{iq} s^{pm} + s^{pn} t^{jq} \\ &= t^{pm} s^{pm} + s^{d-pm} t^{d-pm} \\ &= 1. \end{aligned}$$

Since these elements in $M_d(k)$ satisfy the relations, R is non-trivial. ∎

We note, in the above proof, that

$$b^{i+j} = t^{q(i+j)} = t^{p(m+n)} = t^d = 1.$$

Hence, for those elements $b, f \in M_d(k)$,

$$b^{M(i+j)+i} f^m + f^n b^{N(i+j)+j} = 1$$

for all positive integers M, N. One can see from this that the condition, in Theorem 2.5, that $i/m = j/n$ is not necessary. It would be of interest to have necessary and sufficient conditions upon i, j, m, n for such a non-trivial algebra to exist. Some results in this direction appear in [1].

Next we return to the two element criteria appearing in Theorem 2.2. In these cases, as in Theorem 1.8, the universal k-algebras can be precisely identified. Our notation is varied to avoid confusion.

THEOREM 2.6: *Let k be a noncommutative ring and R be a k-algebra generated by two elements c and f.*

(i) *Suppose that c and f satisfy precisely the relations $f^n = 0$ and $c^{n-1}f^{n-1} + fc = 1$. Then $R \simeq M_n(k[x])$ for some indeterminate x.*

(ii) *Suppose, in addition, the relation $c^n = 0$ is imposed. Then $R \simeq M_n(k)$.*

Proof: (i) As in the proof of Theorem 1.8, one can see that $R = M_n(S)$ for some ring S with $f = e_{21} + e_{32} + \cdots + e_{n,n-1}$ and with c and c^{n-1} taking the forms, in that proof, of b and a respectively. The form of b shows that

$$c = e_{12} + e_{23} + \cdots + e_{n,n-1} + c_1 e_{n1} + c_2 e_{n2} + \cdots + c_n e_{nn}$$

for some $c_i \in S$. It is not difficult to calculate that the $(2, n)$ entry of c^{n-1} is c_n. The form of a shows then that $c_n = 0$. Consideration, in turn, of the $(3, n), (4, n), \ldots, (n-1, n)$ entries of c^{n-1} demonstrates that $c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Thus only c_1 remains unspecified. The fact that the matrices f, as above, and

$$c' = e_{12} + e_{23} + \cdots + e_{n-1,n} + x e_{n1}$$

in $M_n(k[x])$ satisfy the two relations leads, as in the proof of Theorem 1.8, to the desired conclusion.

(ii) For c as above, one checks readily that $c^n = c_1 1$. So in this case $c_1 = 0$ and, in a similar fashion, $R \simeq M_n(k)$. ∎

Note 2.7: Theorem 2.6(ii), together with Theorem 2.2, shows that one can present the $n \times n$ matrix ring over any ring k by means of a set of just two generators f, c and the three relations

$$f^n = 0, \quad c^n = 0, \quad c^{n-1}f^{n-1} + fc = 1,$$

rather than the usual set of n^2 matrix units and their $n^4 + 1$ relations. Furthermore, the elements f and c then have the form

$$f = e_{21} + e_{32} + \cdots + e_{n,n-1}$$

and

$$c = e_{12} + e_{23} + \cdots + e_{n-1,n}$$

One should, perhaps, recall that Albert [2, page 95] demonstrated the existence of a two element generating set for $M_n(F)$ over a sufficiently large field F, and used this, in [3], to show the same for any finite dimensional separable algebra over an infinite field.

3. Application to rings of differential operators

In this section we will examine certain homomorphic images of rings of differential operators to which the previous material is applicable.

Let R be a ring of prime characteristic $p > 0$, and $\delta: R \to R$ be a derivation. Let $R[t, \delta]$ be the corresponding differential operator ring, in which we have $tr = rt + \delta(r)$ for all $r \in R$.

More generally, by induction on m we have

$$t^m r = \sum_{i=0}^{m} \binom{m}{i} \delta^i(r) t^{m-i}. \tag{4}$$

It is well known that a p-th power of a derivation in a ring of characteristic p is also a derivation, so we have a set $\{\delta^{p^n} : n \geq 0\}$ of derivations on R. In fact we note that for $m = p^n$ (4) becomes

$$t^{p^n} r = r t^{p^n} + \delta^{p^n}(r). \tag{5}$$

THEOREM 3.1: *Let $R[t,\delta]$ be as above. If $n > 0$ is an integer such that $1 \in \operatorname{im}(\delta^{p^{n-1}})$ and $\delta^{p^n} = 0$ then $R[t,\delta]/(t^{p^n})$ is a nonzero $p^n \times p^n$ matrix ring.*

Proof: Let $T = t^{p^{n-1}}$ and $\Delta = \delta^{p^{n-1}}$. Since $1 \in \operatorname{im}(\Delta)$ there is an $s \in R$ with $\Delta(s) = 1$ and so $Ts = sT + 1$. We get by induction on k that $\Delta(s^k) = ks^{k-1}$. Hence by (4) the following holds in $R[t,\delta]$, where at the second step we collect terms ending in T and call the sum $s'T$.

$$\begin{aligned} T^{p-1}s^{p-1} &= \sum_{i=0}^{p-1} \binom{p-1}{i} \Delta^i(s^{p-1})T^{p-1-i} \\ &= s'T + \Delta^{p-1}(s^{p-1}) \\ &= s'T + (p-1)! \\ &= s'T - 1 \end{aligned}$$

by Wilson's theorem. We know generally about rings of differential operators that $R[t,\delta]$ is a free left R-module with basis $\{1, t, t^2, \dots\}$. In view of (5) and the relation $\delta^{p^n} = 0$ we have $t^{p^n}r = rt^{p^n}$ for all $r \in R$. Therefore (t^{p^n}) is a proper ideal of $R[t,\delta]$ with factor ring $R[t,\delta]/(t^{p^n})$ free as a left R-module on the basis $\{1, t, t^2, \dots, t^{p^n-1}\}$.

Now in the factor ring $R[t,\delta]/(t^{p^n})$, there are elements t, s, s' satisfying

$$\begin{aligned} s't^{p^{n-1}} + t^{p^n - p^{n-1}}(-s^{p-1}) &= 1 \\ t^{p^n} &= 0. \end{aligned}$$

So, by Theorem 1.7, $R[t,\delta]/(t^{p^n})$ is a $p^n \times p^n$ matrix ring. ∎

Example 3.2: Let $n \geq 0$ be an integer. Consider $R = k[x_1, x_2, \dots, x_{p^{n-1}}]$ where k is a field of characteristic p. Let δ: $R \to R$ be the k-linear derivation on R defined by $\delta(x_i) = x_{i+1}$ for all $i < p^{n-1}$ and $\delta(x_{p^{n-1}}) = 1$. Here $\delta^{p^{n-1}}(x_1) = 1$ and $\delta^{p^n} = 0$, showing that one has instances of Theorem 3.1 with arbitrarily large n.

Note 3.3: Let k be a field of characteristic p and consider the Weyl algebra over k defined as $A_1(k) = k[x][t, \frac{d}{dx}]$. Here $\frac{d}{dx}(x) = 1$ and $(\frac{d}{dx})^p = 0$, so by Theorem 3.1, $A_1(k)/(t^p)$ is a $p \times p$ matrix algebra. In fact it is known that $k[x][t, \frac{d}{dx}]/(t^p) \cong M_p(k[x])$ as k-algebras [6, exercise 2ZF, p.42], so here we have an example of R and δ such that $R[t,\delta]/(t^p) \cong M_p(R)$. This isomorphism does not hold for general R. For example, by applying Theorem 3.1 and counting dimensions over k we see that as k-algebras $k[x]/(x^p)[t, \frac{d}{dx}]/(t^p) \ncong M_p(k)$.

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